Online Appendix
A Note on the Asymptotic Properties of the GLS Estimator in Multivariate Regression with Heteroskedastic and Autocorrelated Errors

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Abstract: This note presents the proofs of the theorems in the main document, along with the lemmas required for their derivation.

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A.1 Proofs of the theorems and lemmas

Lemma 1: *Given Assumption 1 and 2, the following properties hold:*

(i) *The VAR(p) process* (5) *in the main document has a VMA(∞) representation of the form $\mathbf{e}_t = \sum_{s=0}^{\infty} \mathbf{\Psi}_s \boldsymbol{\varepsilon}_{t-s}$, where the sequence $\{\mathbf{\Psi}_s\}_{s=0}^{\infty}$ is absolutely summable, satisfying*

$$\sum_{s=0}^{\infty} |\psi_{ij}^{(s)}| < \infty, \text{ for } i = 1, 2, \dots, N \text{ and } j = 1, 2, \dots, N.$$

(ii) *The autocavariance $\mathbf{\Gamma}_e^{(s)} := \mathbb{E}(\mathbf{e}_t \mathbf{e}_{t-s}')$ exists and is given by*

$$\mathbf{\Gamma}_e^{(s)} = \sum_{v=0}^{\infty} \mathbf{\Psi}_{s+v} \mathbf{\Omega} \mathbf{\Psi}_v', \ \ s = 0, 1, 2, \dots,$$

and the sequence of $\{\mathbf{\Gamma}_e^{(s)}\}_{s=0}^{\infty}$ is absolutely summable.

(iii) $\mathbb{E}|e_{k_1,t_1} e_{k_2,t_2} e_{k_3,t_3} e_{k_4,t_4}| < \infty$ *for $k_1, k_2, k_3, k_4 = 1, \dots, N$ and for all t_1, t_2, t_3, t_4.*

(iv) $(1/T) \sum_{t=1}^{n} \mathbf{e}_t \mathbf{e}_{t-s}' \xrightarrow{p} \mathbf{\Gamma}_e^{(s)}$ *for all s.*

Proof. Here, we prove only (i). For (ii)-(iv), see Hamilton (1994, Proposition 10.2). We begin by rewriting the VAR(p) model in a companion VAR(1) form:

$$\underbrace{\begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_{t-1} \\ \mathbf{e}_{t-2} \\ \vdots \\ \mathbf{e}_{t-p+1} \end{bmatrix}}_{=\mathbf{u}_t} = \underbrace{\begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \mathbf{\Phi}_3 & \cdots & \mathbf{\Phi}_p \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}}_{=\bar{\mathbf{\Phi}}} \underbrace{\begin{bmatrix} \mathbf{e}_{t-1} \\ \mathbf{e}_{t-2} \\ \vdots \\ \mathbf{e}_{t-p+1} \\ \mathbf{e}_{t-p} \end{bmatrix}}_{=\mathbf{u}_{t-1}} + \underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}}_{=\bar{\boldsymbol{\varepsilon}}_t} \tag{A.1}$$

where $\mathbf{u}_t$ and $\mathbf{u}_{t-1}$ are the $Np \times 1$ matrices, $\bar{\mathbf{\Phi}}$ is $Np \times Np$ matrix, and $\bar{\boldsymbol{\varepsilon}}_t$ collects the innovation $\boldsymbol{\varepsilon}_t$ in its first N rows. By successively substituting $\mathbf{u}_{t-1}, \dots, \mathbf{u}_{t-h}$ in the above, we obtain the following equation:

$$\mathbf{u}_t = \bar{\mathbf{\Phi}} \mathbf{u}_{t-1} + \bar{\boldsymbol{\varepsilon}}_t = \bar{\mathbf{\Phi}}^h \mathbf{u}_{t-h} + \sum_{s=0}^{h-1} \bar{\mathbf{\Phi}}^s \bar{\boldsymbol{\varepsilon}}_{t-s}. \tag{A.2}$$

Under Assumption 1, all eigenvalues of $\bar{\mathbf{\Phi}}$ lie strictly inside the unit circle. Consequently, $\bar{\mathbf{\Phi}}^h \mathbf{u}_{t-h} \to 0$ as $h \to \infty$. Hence, $\mathbf{u}_t = \sum_{s=0}^{\infty} \bar{\mathbf{\Phi}}^s \bar{\boldsymbol{\varepsilon}}_{t-s}$. Note that $\mathbf{e}_t$ can be expressed as follows:

$$\mathbf{e}_t = \boldsymbol{C} \mathbf{u}_t = \left(\sum_{s=0}^{\infty} \boldsymbol{C} \bar{\mathbf{\Phi}}^s \bar{\varepsilon}_{t-s} \right) = \sum_{s=0}^{\infty} \left(\boldsymbol{C} \bar{\mathbf{\Phi}}^s \boldsymbol{C}' \right) \boldsymbol{\varepsilon}_{t-s} = \sum_{s=0}^{\infty} \mathbf{\Psi}_s \boldsymbol{\varepsilon}_{t-s} \tag{A.3}$$

where $\boldsymbol{C} = [\mathbf{I}_N, \mathbf{0}, \dots, \mathbf{0}]$ is $N \times Np$ and $\mathbf{\Psi}_s = \boldsymbol{C} \bar{\mathbf{\Phi}}^s \boldsymbol{C}'$. To prove $\{\mathbf{\Psi}_s\}$ is absolutely absolutely summable, we use any induced matrix norm $\| \cdot \|$ that satisfies submultiplicativity. Observe that

$$\|\mathbf{\Psi}_s\| = \|\boldsymbol{C} \bar{\mathbf{\Phi}}^s \boldsymbol{C}'\| \le \|\boldsymbol{C}\| \ \|\bar{\mathbf{\Phi}}^s\| \ \|\boldsymbol{C}'\|.$$

Since Assumption 1 implies that the spectral radius of $\bar{\mathbf{\Phi}}^s$ is less than 1, there exist $M > 0$ and $0 < r < 1$ such that $\|\bar{\mathbf{\Phi}}^s\| \leq Mr^s$ for all $s \geq 0$. Therefore,

$$\sum_{s=0}^{\infty} \|\mathbf{\Psi}_s\| \leq \|\boldsymbol{C}\| \, \|\boldsymbol{C}'\| M \sum_{s=0}^{\infty} r^s \leq \|\boldsymbol{C}\| \, \|\boldsymbol{C}'\| M \frac{1}{1-r} < \infty.$$

Hence, $\mathbf{\Psi}_s$ is absolutely summable. □

Lemma 2: *Under the Assumptions 1 and 2, then*

$$\frac{1}{\sqrt{T}} \sum_{i=1}^{T} \boldsymbol{e}_t \xrightarrow{d} \mathcal{N}(\mathbf{0}_{N,1}, \mathbf{\Gamma}_e^{\infty}),$$

where $\mathbf{\Gamma}_e^{\infty} := \sum_{s=-\infty}^{\infty} \mathbf{\Gamma}_e^{(s)}$.

Proof. From Lemma 1, the VAR(p) process (5) in the main document can be expressed as the following VMA(∞) representation:

$$\boldsymbol{e}_t = \sum_{s=0}^{\infty} \mathbf{\Psi}_s \boldsymbol{\varepsilon}_{t-s}, \tag{A.4}$$

where $\sum_{s=0}^{\infty} |\psi_{ij}^{(s)}| < \infty$, for $i = 1, 2, \ldots, N$ and$j = 1, 2, \ldots, N$. Define $\bar{\boldsymbol{e}} = 1/T \sum_{t=1}^{T} \boldsymbol{e}_t$. Then we have $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{e}_t = \sqrt{T}\bar{\boldsymbol{e}}$. Since the vector sequence of $\boldsymbol{\varepsilon}_t$ has zero mean, the variance of the sample mean can be written from Lemma 1 as follows:

$$\begin{aligned}
\mathbb{E}(\bar{\boldsymbol{e}}\bar{\boldsymbol{e}}') =& \frac{1}{T^2} \mathbb{E}[(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_T)(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_T)'] \\
=& \frac{1}{T^2} \mathbb{E}[\boldsymbol{e}_1(\boldsymbol{e}_1' + \boldsymbol{e}_2' + \cdots + \boldsymbol{e}_T') + \boldsymbol{e}_2(\boldsymbol{e}_1' + \boldsymbol{e}_2' + \cdots + \boldsymbol{e}_T') \\
& + \boldsymbol{e}_3(\boldsymbol{e}_1' + \boldsymbol{e}_2' + \cdots + \boldsymbol{e}_T') + \cdots + \boldsymbol{e}_T(\boldsymbol{e}_1' + \boldsymbol{e}_2' + \cdots + \boldsymbol{e}_T')] \\
=& \frac{1}{T^2} [(\mathbf{\Gamma}_{e,0} + \mathbf{\Gamma}_{e,-1} + \cdots + \mathbf{\Gamma}_{e,-(T-1)}) \\
& + (\mathbf{\Gamma}_{e,1} + \mathbf{\Gamma}_{e,0} + \mathbf{\Gamma}_{e,-1} + \cdots + \mathbf{\Gamma}_{e,-(T-2)}) \\
& + (\mathbf{\Gamma}_{e,2} + \mathbf{\Gamma}_{e,1} + \mathbf{\Gamma}_{e,0} + \cdots + \mathbf{\Gamma}_{e,-(T-3)}) \\
& + \cdots + (\mathbf{\Gamma}_{e,T-1} + \mathbf{\Gamma}_{e,T-2} + \mathbf{\Gamma}_{e,T-3} + \cdots + \mathbf{\Gamma}_{e,0})] \\
=& \frac{1}{T^2} [T\mathbf{\Gamma}_{e,0} + (T-1)\mathbf{\Gamma}_{e,1} + (T-2)\mathbf{\Gamma}_{e,2} + \cdots + \mathbf{\Gamma}_{e,T-1} \\
& + (T-1)\mathbf{\Gamma}_{e,-1} + (T-1)\mathbf{\Gamma}_{e,-2} + \cdots + \mathbf{\Gamma}_{e,-(T-1)}].
\end{aligned} \tag{A.5}$$

Thus, we have

$$\begin{aligned}
T\mathbb{E}(\bar{\boldsymbol{e}}\bar{\boldsymbol{e}}') =& [\mathbf{\Gamma}_e^{(0)} + \frac{T-1}{T} \mathbf{\Gamma}_e^{(1)} + \frac{T-2}{T} \mathbf{\Gamma}_e^{(2)} + \cdots + \frac{1}{T} \mathbf{\Gamma}_e^{(T-1)} \\
& + \frac{T-1}{T} \mathbf{\Gamma}_e^{(-1)} + \frac{T-2}{T} \mathbf{\Gamma}_e^{(-2)} + \cdots + \frac{1}{T} \mathbf{\Gamma}_e^{(-(T-1))}].
\end{aligned} \tag{A.6}$$

Consider the following matrix:

$$\sum_{v=-\infty}^{\infty} \mathbf{\Gamma}_e^{(v)} - T\mathbb{E}(\bar{\boldsymbol{e}}\bar{\boldsymbol{e}}') = \sum_{|v| \geq T} \mathbf{\Gamma}_e^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \mathbf{\Gamma}_e^{(v)}. \tag{A.7}$$

Let $\gamma_{e,ij}^{(v)}$ denote the row i, column j element of $\mathbf{\Gamma}_e^{(v)}$. The row i, column j element of the matrix in (A.7) ca be written

$$\left| \sum_{|v| \geq T} \gamma_{e,ij}^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \gamma_{e,ij}^{(v)} \right| \leq \sum_{|v| \geq T} \left| \gamma_{e,ij}^{(v)} \right| + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \left| \gamma_{e,ij}^{(v)} \right|. \tag{A.8}$$

Since $\{\gamma_{e,ij}^{(v)}\}$ is absolutely summable from Lemma 1, for any $\varepsilon > 0$ there exists a p such that $\sum_{|v|>p} |\gamma_{e,ij}^{(v)}| < \varepsilon/2$. Therefore,

$$\left| \sum_{|v| \geq T} \gamma_{e,ij}^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \gamma_{e,ij}^{(v)} \right| < \frac{\varepsilon}{2} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \left| \gamma_{e,ij}^{(v)} \right|. \tag{A.9}$$

Moreover, for this given p, we can find an u such that$\sum_{v=-(T-1)}^{T-1} |v|/T |\gamma_{e,ij}^{(v)}| < \varepsilon/2$ for all $T > u$, ensuring that

$$\left| \sum_{|v| \geq T} \gamma_{e,ij}^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \gamma_{e,ij}^{(v)} \right| < \varepsilon. \tag{A.10}$$

This result holds for each i and j. Therefore, we have

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{e}_t \xrightarrow{d} \mathcal{N}\left(\mathbf{0}_{N,1}, \sum_{v=-\infty}^{\infty} \mathbf{\Gamma}_e^{(v)} \right). \tag{A.11}$$

□

Lemma 3: *Given Assumption 2,* $\mathbf{\Gamma}_V := \operatorname*{plim}_{T\to\infty} \mathbf{V}\mathbf{V}'/(T-p)$ *exists and nonsingular,* $\widehat{\mathbf{\Phi}} \xrightarrow{p} \mathbf{\Phi}$, $\widehat{\mathbf{\Omega}} \xrightarrow{p} \mathbf{\Omega}$, *and* $\sqrt{T-p}(\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{N^2p,1}, \mathbf{\Gamma}_V^{-1} \otimes \mathbf{\Omega})$.

Proof. See Lütkepohl (2005, Proposition 3.1). □

Proof of Theorem 1

We define

$$\widehat{\mathbf{M}}_Z := \frac{\mathbf{Z}'\mathbf{Z}}{T} = \frac{1}{T} \begin{bmatrix} \mathbf{E}_{T,N}' \\ \mathbf{X}' \end{bmatrix} \begin{bmatrix} \mathbf{E}_{T,N} & \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_N & \bigoplus_{i=1}^N (\frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t}') \\ \bigoplus_{i=1}^N (\frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t}) & \bigoplus_{i=1}^N (\frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t}\mathbf{x}_{i,t}') \end{bmatrix}.$$

By Assumption 3 (i), $\widehat{\mathbf{M}}_Z \xrightarrow{p} \mathbf{M}_Z$. Here, the symbol $\bigoplus$ denotes the direct sum of matrices. Thus, we have

$$\bigoplus_{i=1}^N \left(\frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t} \right) \xrightarrow{p} \boldsymbol{\mu}_X := \bigoplus_{i=1}^N \boldsymbol{\mu}_{x,i} \text{ and } \bigoplus_{i=1}^N \left(\frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t}\mathbf{x}_{i,t}' \right) \xrightarrow{p} \mathbf{\Gamma}_X^{(0)} := \bigoplus_{i=1}^N \mathbf{\Gamma}_{x,ii}^{(0)}, \tag{A.12}$$

where

$$\boldsymbol{\mu}_X = \begin{bmatrix} \boldsymbol{\mu}_{x,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu}_{x,2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\mu}_{x,N} \end{bmatrix} \text{ and } \mathbf{\Gamma}_X^{(0)} = \begin{bmatrix} \mathbf{\Gamma}_{x,11}^{(0)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_{x,22}^{(0)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Gamma}_{x,NN}^{(0)} \end{bmatrix}.$$

It follows from Assumption 3 that $\boldsymbol{\Gamma}_X^{(0)}$ is positive definite. Consequently, $\widehat{\mathbf{M}}_Z$ is also positive definite, ensuring the existence of the OLS estimator:

$$\widehat{\boldsymbol{\kappa}}_T^{OLS} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{e}) = \boldsymbol{\kappa}_0 + \left(\frac{\mathbf{Z}'\mathbf{Z}}{T}\right)^{-1}\frac{\mathbf{Z}'\mathbf{e}}{T} = \boldsymbol{\kappa}_0 + \widehat{\mathbf{M}}_Z^{-1}\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}_t, \tag{A.13}$$

where $\mathbf{w}_t = [\mathbf{e}_t', \mathbf{e}_t'\mathbf{X}_t]'$. Define $\boldsymbol{\Gamma}_{w,s} := \mathbb{E}(\mathbf{w}_t\mathbf{w}_{t-s}')$ as

$$\begin{aligned}
\boldsymbol{\Gamma}_w^{(s)} =& \mathbb{E}(\mathbf{w}_t\mathbf{w}_{t-s}') \qquad \text{(A.14)}\\
=& E\left(\begin{bmatrix}\mathbf{e}_t\\ \mathbf{X}_t'\mathbf{e}_t\end{bmatrix}\begin{bmatrix}\mathbf{e}_{t-s}' & \mathbf{e}_{t-s}'\mathbf{X}_{t-s}\end{bmatrix}\right)\\
=& \begin{bmatrix}\mathbb{E}(\mathbf{e}_t\mathbf{e}_{t-s}') & \mathbb{E}(\mathbf{e}_t\mathbf{e}_{t-s}'\mathbf{X}_{t-s}')\\ \mathbb{E}(\mathbf{X}_t'\mathbf{e}_t\mathbf{e}_{t-s}') & \mathbb{E}(\mathbf{X}_t'\mathbf{e}_t\mathbf{e}_{t-s}'\mathbf{X}_{t-s})\end{bmatrix}\\
=& \begin{bmatrix}\mathbb{E}(\mathbf{e}_t\mathbf{e}_{t-s}') & \mathbb{E}(\mathbf{e}_t\mathbf{e}_{t-s}')\mathbb{E}(\mathbf{X}_{t-s}')\\ \mathbb{E}(\mathbf{X}_t')\mathbb{E}(\mathbf{e}_t\mathbf{e}_{t-s}') & [\mathbb{E}(\mathbf{x}_{i,t}\mathbf{x}_{j,t-s}')\mathbb{E}(e_{i,t}e_{j,t-s})]_{i,j=1}^N\end{bmatrix}\\
=& \begin{bmatrix}\boldsymbol{\Gamma}_e^{(s)} & \boldsymbol{\Gamma}_e^{(s)}\boldsymbol{\mu}_X'\\ \boldsymbol{\mu}_X\boldsymbol{\Gamma}_e^{(s)} & [\boldsymbol{\Gamma}_{x,ij}^{(s)}\gamma_{e,ij}^{(s)}]_{i,j=1}^N\end{bmatrix}. \qquad \text{(A.15)}
\end{aligned}$$

Note that $[\boldsymbol{\Gamma}_{x,ij}^{(s)}\gamma_{e,ij}^{(s)}]_{i,j=1}^N$ is a $N\times N$ block matrix, where (i,j)-th block is given by $\boldsymbol{\Gamma}_{x,ij}^{(s)}\gamma_{e,ij}^{(s)}$. Specifically, $[\boldsymbol{\Gamma}_{x,ij}^{(s)}\gamma_{e,ij}^{(s)}]_{i,j=1}^N$ can be expressed as:

$$\begin{aligned}
&\mathbb{E}(\mathbf{X}_t'\mathbf{e}_t\mathbf{e}_{t-s}'\mathbf{X}_{t-s})\\
=&\begin{bmatrix}
\mathbb{E}(\mathbf{x}_{1,t}\mathbf{x}_{1,t-s}')\mathbb{E}(e_{1,t}e_{1,t-s}) & \mathbb{E}(\mathbf{x}_{1,t}\mathbf{x}_{1,t-s}')\mathbb{E}(e_{1,t}e_{2,t-s}) & \cdots & \mathbb{E}(\mathbf{x}_{1,t}\mathbf{x}_{N,t-s}')\mathbb{E}(e_{1,t}e_{N,t-s})\\
\mathbb{E}(\mathbf{x}_{2,t}\mathbf{x}_{2,t-s}')\mathbb{E}(e_{2,t}e_{1,t-s}) & \mathbb{E}(\mathbf{x}_{2,t}\mathbf{x}_{2,t-s}')\mathbb{E}(e_{2,t}e_{2,t-s}) & \cdots & \mathbb{E}(\mathbf{x}_{2,t}\mathbf{x}_{N,t-s}')\mathbb{E}(e_{2,t}e_{N,t-s})\\
\vdots & \vdots & \ddots & \vdots\\
\mathbb{E}(\mathbf{x}_{N,t}\mathbf{x}_{1,t-s}')\mathbb{E}(e_{N,t}e_{1,t-s}) & \mathbb{E}(\mathbf{x}_{2,t}\mathbf{x}_{N,t-s}')\mathbb{E}(e_{2,t}e_{N,t-s}) & \cdots & \mathbb{E}(\mathbf{x}_{N,t}\mathbf{x}_{N,t-s}')\mathbb{E}(e_{N,t}e_{N,t-s})
\end{bmatrix}\\
=&\begin{bmatrix}
\boldsymbol{\Gamma}_{x,11}^{(s)}\gamma_{e,11}^{(s)} & \boldsymbol{\Gamma}_{x,12}^{(s)}\gamma_{e,12}^{(s)} & \cdots & \boldsymbol{\Gamma}_{x,1N}^{(s)}\gamma_{e,1N}^{(s)}\\
\boldsymbol{\Gamma}_{x,21}^{(s)}\gamma_{e,21}^{(s)} & \boldsymbol{\Gamma}_{x,22}^{(s)}\gamma_{e,22}^{(s)} & \cdots & \boldsymbol{\Gamma}_{x,2N}^{(s)}\gamma_{e,2N}^{(s)}\\
\vdots & \vdots & \ddots & \vdots\\
\boldsymbol{\Gamma}_{X,N1}^{(s)}\gamma_{e,N1}^{(s)} & \boldsymbol{\Gamma}_{x,N2}^{(s)}\gamma_{e,N2}^{(s)} & \cdots & \boldsymbol{\Gamma}_{x,NN}^{(s)}\gamma_{e,NN}^{(s)}
\end{bmatrix}. \qquad \text{(A.16)}
\end{aligned}$$

Let $\boldsymbol{\Gamma}_e^{\infty} = \sum_{s=-\infty}^{\infty}\boldsymbol{\Gamma}_e^{(s)}$ and $\boldsymbol{\Gamma}_w^{\infty} = \sum_{s=-\infty}^{\infty}\boldsymbol{\Gamma}_w^{(s)}$. It follows from Lemma 1 that $\boldsymbol{\Gamma}_e^{\infty} < \infty$, and from Lemma 2 that

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\mathbf{w}_t \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}_w^{\infty}), \tag{A.17}$$

where

$$\boldsymbol{\Gamma}_w^{\infty} := \begin{bmatrix}\sum_{s=-\infty}^{\infty}\boldsymbol{\Gamma}_e^{(s)} & \sum_{s=-\infty}^{\infty}\boldsymbol{\Gamma}_e^{(s)}\boldsymbol{\mu}_X'\\ \boldsymbol{\mu}_X\sum_{s=-\infty}^{\infty}\boldsymbol{\Gamma}_e^{(s)} & \sum_{s=-\infty}^{\infty}[\boldsymbol{\Gamma}_{x,ij}^{(s)}\gamma_{e,ij}^{(s)}]_{i,j=1}^N\end{bmatrix} < \infty, \tag{A.18}$$

Then, we can apply the multivariate Lindeberg-Lévy central limit theorem. Therefore, we have

$$\sqrt{T}(\widehat{\boldsymbol{\kappa}}_T^{OLS} - \boldsymbol{\kappa}_0) = \widehat{\mathbf{M}}_Z^{-1}T^{-1/2}\sum_{t=1}^{T}\mathbf{w}_t \xrightarrow{d} \mathcal{N}(\mathbf{0}_{(1+K)N,1}, \mathbf{M}_Z^{-1}\boldsymbol{\Gamma}_w^{\infty}\mathbf{M}_Z^{-1}). \tag{A.19}$$

□

Proof of Theorem 2

From Lemma 3, we obtain consistent estimators for $\mathbf{\Phi}$, $\mathbf{\Omega}$, and $\mathbf{\Gamma}_e^{\infty}$. We define

$$\mathbf{M}_Z^{QD} := \mathbb{E}(\mathbf{Z}_t^{QD\prime}\mathbf{\Omega}^{-1}\mathbf{Z}_t^{QD}), \quad \text{and} \quad \mathbf{M}_Z^{QD_2} := \mathbb{E}(\mathbf{Z}_t^{QD_2\prime}\mathbf{\Omega}^{-1}\mathbf{Z}_t^{QD_2}). \tag{A.20}$$

Then we have,

$$\begin{aligned}
\widehat{\mathbf{M}}_Z^{QD} =& \frac{\widehat{\mathbf{Z}}^{QD\prime}(\mathbf{I}_T \otimes \widehat{\mathbf{\Omega}}^{-1})\widehat{\mathbf{Z}}^{QD}}{T} \\
=& \frac{1}{T}\begin{bmatrix}\widehat{\mathbf{Z}}^{QD_1\prime} & \widehat{\mathbf{Z}}^{QD_2\prime}\end{bmatrix}(\mathbf{I}_T \otimes \widehat{\mathbf{\Omega}}^{-1})\begin{bmatrix}\widehat{\mathbf{Z}}^{QD_1} \\ \widehat{\mathbf{Z}}^{QD_2}\end{bmatrix} \\
=& \frac{1}{T}\left[\widehat{\mathbf{Z}}^{QD_1\prime}(\mathbf{I}_p \otimes \widehat{\mathbf{\Omega}}^{-1})\widehat{\mathbf{Z}}^{QD_1} + \widehat{\mathbf{Z}}^{QD_2\prime}(\mathbf{I}_{T-p} \otimes \widehat{\mathbf{\Omega}}^{-1})\widehat{\mathbf{Z}}^{QD_2}\right] \\
=& \frac{1}{T}\left[\textstyle\sum_{t=1}^{p} \widehat{\mathbf{Z}}_t^{QD_1\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\mathbf{Z}}_t^{QD_1} + \sum_{t=1}^{T-p} \widehat{\mathbf{Z}}_t^{QD_2\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\mathbf{Z}}_t^{QD_2}\right] \\
\xrightarrow{p}& \mathbb{E}(\mathbf{Z}_t^{QD_2\prime}\mathbf{\Omega}^{-1}\mathbf{Z}_t^{QD_2}).
\end{aligned} \tag{A.21}$$

Therefore, $\widehat{\mathbf{M}}_Z^{QD} \xrightarrow{p} \mathbf{M}_Z^{QD_2}$. Moreover, $\mathbf{M}_Z^{QD_2}$ is invertible. Then we have

$$\begin{aligned}
\widehat{\boldsymbol{\kappa}}_n^{PW} =& (\widehat{\mathbf{Z}}^{QD\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\mathbf{Z}}^{QD})^{-1}\widehat{\mathbf{Z}}^{QD\prime}\widehat{\mathbf{\Omega}}^{-1}(\widehat{\mathbf{Z}}^{QD}\boldsymbol{\kappa}_0 + \widehat{\boldsymbol{\varepsilon}}^{QD}) \\
=& \boldsymbol{\kappa}_0 + \left(\frac{\widehat{\mathbf{Z}}^{QD\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\mathbf{Z}}^{QD}}{T}\right)^{-1}\frac{\widehat{\mathbf{Z}}^{QD\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\boldsymbol{\varepsilon}}^{QD}}{T} \\
=& \boldsymbol{\kappa}_0 + \widehat{\mathbf{M}}_Z^{QD-1}\frac{1}{T}\left(\sum_{t=1}^{p}\widehat{\mathbf{w}}_t^{QD_1} + \sum_{t=1}^{T-p}\widehat{\mathbf{w}}_t^{QD_2}\right).
\end{aligned} \tag{A.22}$$

where $\widehat{\mathbf{w}}_t^{QD_1} = \widehat{\mathbf{Z}}_t^{QD_1\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\boldsymbol{\varepsilon}}_t^{QD_1}$, and $\widehat{\mathbf{w}}_t^{QD_2} = \widehat{\mathbf{Z}}_t^{QD_2\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\boldsymbol{\varepsilon}}_t^{QD_2}$. Under Assumption 2, we have

$$\frac{1}{\sqrt{T}}\left(\sum_{t=1}^{p}\widehat{\mathbf{w}}_t^{QD_1} + \sum_{t=1}^{T-p}\widehat{\mathbf{w}}_t^{QD_2}\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{(N+K),1}, \mathbf{M}_Z^{QD_2}). \tag{A.23}$$

Thus we have

$$\sqrt{T}(\widehat{\boldsymbol{\kappa}}_T^{PW} - \boldsymbol{\kappa}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{(N+K),1}, \mathbf{M}_Z^{QD_2-1}). \tag{A.24}$$

□

Proof of Theorem 3

From Theorem 2, we have the asymptotic distribution:

$$\sqrt{T}(\widehat{\boldsymbol{\kappa}}_T^{PW} - \boldsymbol{\kappa}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{(N+K),1}, \mathbf{M}_Z^{QD_2-1}). \tag{A.25}$$

Under the null hypothesis $H_0 : \mathbf{R}\boldsymbol{\kappa}_0 = \mathbf{r}$, it follows from (A.25) that:

$$\sqrt{T}(\mathbf{R}\widehat{\boldsymbol{\kappa}}_T^{PW} - \mathbf{r}) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{r,1}, \mathbf{R}\mathbf{M}_Z^{QD_2-1}\mathbf{R}'). \tag{A.26}$$

Additionally, since $\widehat{\mathbf{M}}_Z^{QD} = \frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{Z}}_t^{QD\prime}\widehat{\mathbf{\Omega}}^{-1}\widehat{\mathbf{Z}}_t^{QD} \xrightarrow{p} \mathbf{M}_Z^{QD_2}$, we have:

$$T(\mathbf{R}\widehat{\boldsymbol{\kappa}}_T^{PW} - \mathbf{r})'(\mathbf{R}\widehat{\mathbf{M}}_Z^{QD-1}\mathbf{R}')^{-1}(\mathbf{R}\widehat{\boldsymbol{\kappa}}_T^{PW} - \mathbf{r}) \xrightarrow{d} \chi_r^2, \tag{A.27}$$

where χ_r^2 represents the chi-squared distribution with r degrees of freedom. □

A.2 Rejection rates under the alternative hypothesis

(Table A.1 and A.2 around here)

Tables A.1 and A.2 present the empirical power (Type II error rates) of the tests for multiple regression under (i) heteroskedasticity and (ii) both heteroskedasticity and autocorrelation. All tests exhibit reasonably strong power regardless of the error term process. Notably, our proposed Wald tests, designed under the assumption that the error terms follow a VAR process, demonstrate high test power in both cases.

References

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Table A.1: Rejection rates under the alternative hypothesis $H_1 : \boldsymbol{\alpha}_0 \neq \mathbf{0}_{N,1}$ (heteroskedastic errors)

		$\mathcal{W}^{PW}$			$\mathcal{W}^{CO}$			$\mathcal{W}^{HAR}$			GRS			GRS^{KS}		
		10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 6/K = 3$																
	$T = 200$	0.402	0.273	0.131	0.390	0.268	0.130	0.421	0.310	0.156	0.312	0.209	0.064	0.312	0.209	0.064
	$T = 400$	0.609	0.495	0.294	0.609	0.494	0.294	0.632	0.510	0.320	0.568	0.433	0.247	0.568	0.433	0.247
	$T = 800$	0.881	0.815	0.626	0.878	0.813	0.629	0.883	0.826	0.648	0.872	0.801	0.597	0.872	0.801	0.597
	$T = 1600$	0.990	0.981	0.945	0.990	0.980	0.945	0.991	0.981	0.942	0.989	0.979	0.943	0.989	0.979	0.943
	$T = 3200$	1.000	1.000	0.998	1.000	1.000	0.998	1.000	1.000	0.998	1.000	1.000	0.998	1.000	1.000	0.998
$N = 6/K = 5$																
	$T = 200$	0.440	0.323	0.152	0.435	0.324	0.151	0.476	0.355	0.192	0.326	0.208	0.083	0.326	0.208	0.082
	$T = 400$	0.609	0.506	0.279	0.609	0.507	0.285	0.613	0.523	0.308	0.570	0.444	0.232	0.570	0.444	0.231
	$T = 800$	0.899	0.832	0.628	0.897	0.834	0.627	0.895	0.838	0.649	0.884	0.822	0.603	0.884	0.822	0.603
	$T = 1600$	0.992	0.981	0.938	0.991	0.980	0.940	0.992	0.982	0.945	0.990	0.981	0.933	0.990	0.981	0.933
	$T = 3200$	1.000	1.000	0.999	1.000	1.000	0.999	1.000	1.000	0.999	1.000	1.000	0.999	1.000	1.000	0.999
$N = 25/K = 3$																
	$T = 200$	0.761	0.669	0.498	0.770	0.663	0.493	0.844	0.781	0.629	0.230	0.134	0.035	0.230	0.134	0.035
	$T = 400$	0.645	0.534	0.340	0.645	0.530	0.338	0.735	0.647	0.460	0.359	0.238	0.086	0.359	0.238	0.086
	$T = 800$	0.787	0.673	0.464	0.788	0.675	0.462	0.841	0.748	0.559	0.658	0.525	0.301	0.658	0.525	0.301
	$T = 1600$	0.961	0.925	0.823	0.962	0.925	0.827	0.969	0.944	0.851	0.948	0.899	0.763	0.948	0.899	0.763
	$T = 3200$	1.000	1.000	0.991	1.000	1.000	0.991	1.000	0.999	0.996	1.000	1.000	0.991	1.000	1.000	0.991
$N = 25/K = 5$																
	$T = 200$	0.771	0.686	0.520	0.769	0.689	0.514	0.873	0.804	0.670	0.227	0.135	0.034	0.227	0.135	0.034
	$T = 400$	0.638	0.537	0.336	0.635	0.535	0.335	0.752	0.660	0.464	0.359	0.239	0.080	0.358	0.239	0.080
	$T = 800$	0.787	0.688	0.463	0.791	0.684	0.459	0.835	0.762	0.563	0.670	0.529	0.301	0.670	0.529	0.301
	$T = 1600$	0.955	0.922	0.815	0.954	0.921	0.818	0.969	0.941	0.851	0.933	0.895	0.767	0.933	0.894	0.767
	$T = 3200$	0.999	0.999	0.986	0.999	0.999	0.984	0.999	0.999	0.990	0.999	0.998	0.983	0.999	0.998	0.983

Note: R version 4.4.2 was used to compute the statistics.

Table A.2: Rejection rates under the alternative hypothesis $H_1 : \boldsymbol{\alpha}_0 \neq \mathbf{0}_{N,1}$ (heteroskedastic and autocorrelated errors)

		$\mathcal{W}^{PW}$			$\mathcal{W}^{CO}$			$\mathcal{W}^{HAR}$			GRS			GRS^{KS}		
		10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 6/K = 3$																
	$T = 200$	0.313	0.212	0.092	0.312	0.206	0.093	0.400	0.298	0.142	0.575	0.453	0.274	0.575	0.453	0.274
	$T = 400$	0.407	0.303	0.144	0.406	0.302	0.143	0.486	0.384	0.202	0.726	0.643	0.434	0.726	0.643	0.434
	$T = 800$	0.612	0.490	0.268	0.617	0.494	0.263	0.683	0.583	0.363	0.882	0.829	0.682	0.882	0.829	0.682
	$T = 1600$	0.881	0.808	0.606	0.880	0.808	0.605	0.899	0.847	0.681	0.980	0.966	0.917	0.980	0.966	0.917
	$T = 3200$	0.987	0.975	0.922	0.987	0.974	0.921	0.991	0.981	0.937	0.998	0.997	0.994	0.998	0.997	0.994
$N = 6/K = 5$																
	$T = 200$	0.367	0.252	0.118	0.363	0.253	0.116	0.453	0.344	0.178	0.598	0.507	0.288	0.597	0.507	0.288
	$T = 400$	0.416	0.299	0.151	0.412	0.286	0.155	0.506	0.390	0.207	0.715	0.616	0.428	0.715	0.616	0.428
	$T = 800$	0.633	0.500	0.284	0.625	0.501	0.285	0.705	0.588	0.366	0.900	0.854	0.705	0.900	0.854	0.705
	$T = 1600$	0.876	0.797	0.611	0.879	0.798	0.610	0.903	0.843	0.678	0.974	0.958	0.914	0.974	0.958	0.914
	$T = 3200$	0.992	0.979	0.929	0.992	0.980	0.928	0.996	0.987	0.952	0.999	0.999	0.997	0.999	0.999	0.997
$N = 25/K = 3$																
	$T = 200$	0.866	0.810	0.639	0.865	0.811	0.645	0.917	0.890	0.787	0.861	0.789	0.571	0.861	0.789	0.571
	$T = 400$	0.636	0.537	0.317	0.628	0.540	0.319	0.777	0.696	0.511	0.889	0.833	0.663	0.889	0.833	0.663
	$T = 800$	0.596	0.480	0.259	0.596	0.477	0.261	0.742	0.645	0.434	0.958	0.932	0.813	0.958	0.932	0.813
	$T = 1600$	0.749	0.668	0.434	0.752	0.667	0.435	0.859	0.784	0.604	0.992	0.980	0.942	0.992	0.980	0.942
	$T = 3200$	0.947	0.907	0.789	0.947	0.904	0.787	0.970	0.949	0.861	1.000	0.999	0.997	1.000	0.999	0.997
$N = 25/K = 5$																
	$T = 200$	0.865	0.823	0.678	0.874	0.821	0.673	0.933	0.906	0.823	0.837	0.757	0.565	0.837	0.757	0.563
	$T = 400$	0.632	0.523	0.335	0.636	0.521	0.338	0.788	0.705	0.528	0.886	0.831	0.658	0.886	0.831	0.658
	$T = 800$	0.603	0.492	0.271	0.601	0.491	0.274	0.758	0.659	0.458	0.953	0.921	0.823	0.953	0.921	0.823
	$T = 1600$	0.758	0.670	0.444	0.758	0.672	0.447	0.856	0.792	0.604	0.995	0.985	0.944	0.995	0.985	0.944
	$T = 3200$	0.949	0.913	0.808	0.949	0.913	0.809	0.973	0.943	0.881	1.000	1.000	0.996	1.000	1.000	0.996

Note: R version 4.4.2 was used to compute the statistics.